Locally Linear Discriminant Analysis for Multi-modally Distributed Classes For Face Recognition with a Single Model Image

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Abstract– We present a novel method of learning a set of locally linear transformations called "Locally Linear Discriminant Analysis (LLDA)" for nonlinear classification problems. The underlying idea is that global nonlinear data structures are locally linear and local structures can be linearly aligned. Input vectors are projected into each local feature space by linear transformations found to yield locally linearly transformed classes that maximize the between-class covariance while minimizing the within-class covariance in the aligned output space. This maximizes the separability of classes locally while promoting consistency between the multiple local representations of single class objects. In face recognition, linear discriminant analysis (LDA) has been widely adopted owing to its efficiency but it does not capture nonlinear manifolds of faces which exhibit pose variations. Conventional kernel-based nonlinear classification methods such as generalized discriminant analysis (GDA) and support vector machine (SVM) classification have the drawbacks of high computational cost and potential overfitting. Our method is suitable for multi-class nonlinear discrimination and it is highly computationally efficient compared to GDA. Due to the linear base structure of the solution the method does not suffer from overfitting. A novel gradient based learning algorithm is proposed for finding the optimal set of local linear bases. The optimization does not exhibit the problem of local maxima. The *discriminative and aligned* transformation functions facilitate robust face recognition in a low dimensional subspace under pose variations using a single model image. The classification results are given for both synthetic and real face data.

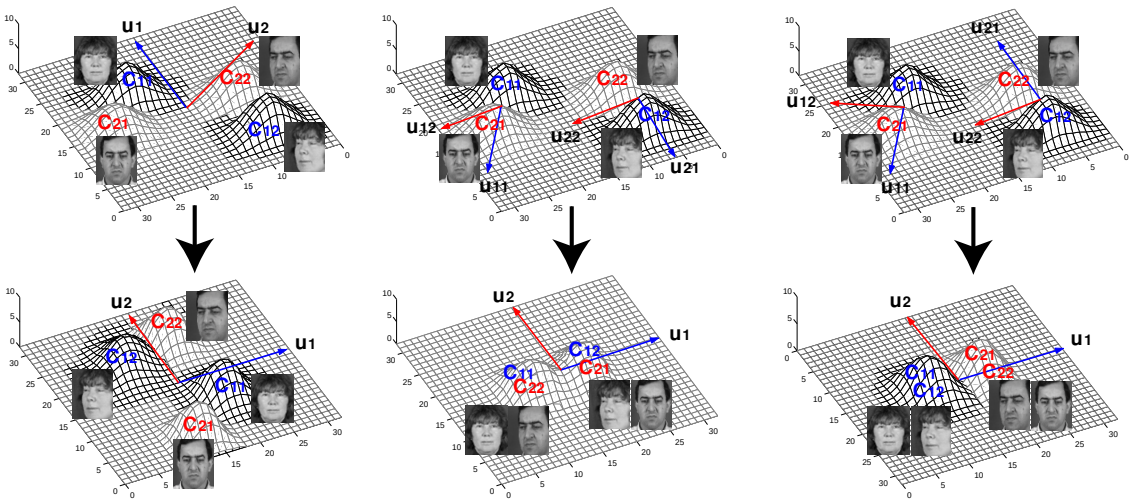
Keywords– Linear Discriminant Analysis, Generalized Discriminant Analysis, Support Vector Machine, Dimensionality Reduction, Face Recognition, Feature Extraction, Pose Invariance, Subspace Representation

1 Introduction

The effectiveness of pattern classification methods can seriously be compromised by various factors which often affect sensory information about an object. Frequently observations from a single object class are multi-modally distributed and samples of objects from different classes in the original data space are more closely located to each other than to those of the same class. The data set of face images taken from a certain number of different viewing angles is a typical example of such problems. It is because the appearance change of face images due to pose changes is usually larger than that caused by different identities. Generally, the face manifold is known to be continuous with respect to continuous pose changes in [23]. The proposed method for multi-modally distributed face classes may be useful generally, as a continuous pose set can be divided into many subsets of multi-modal distributions.

Linear Discriminant Analysis (LDA) [8, 20, 21] is a powerful method for face recognition yielding an effective representation that linearly transforms the original data space into a low dimensional feature space where the data is as well separated as possible under the assumption that the data classes are gaussian with equal co-variance structure. However, the method fails to solve non-linear problems as illustrated in Figure 1 (a), because LDA only considers a single linear transformation in a global coordinate system. The transformed face classes are still multi-modally distributed. The multiple LDA system [7, 16, 25] which adopts several independent local transformations attempts to overcome the shortcomings of LDA but it fails to learn any global data structure as shown in Figure 1 (b). In the LDA mixture model [7, 16], it is assumed that single class objects are distributed normally with an identity covariance matrix structure. Then it just focuses on maximizing the discriminability of the local structures and it does not make any effort to achieve consistency of the local representations for any single object class. In the upper picture of Figure 1 (b), the two data sets C_{11} and C_{12} corresponding to the different modalities of a class are unfortunately positioned in different directions of the corresponding local components, u_{11} and u_{21}, therefore having different representations in a global coordinates as illustrated below. Different classes are mixed up in the transformed space. The view-based method for face recognition proposed by Pentland [25] would experience the same difficulty in these circumstances. Following their idea, we could divide images into different pose groups and then train LDA separately for each group, which is similar to using the LDA mixture. Because these LDA bases do not encode any relationships of the different pose groups, it is not guaranteed that this 'view-based LDA' would yield a consistent representation of different pose images of a single identity. In many conventional face recognition systems [7, 18, 20, 21, 25] which adopt a linear machine such as LDA or LDA mixture model, as many gallery samples as possible are required so as to capture all the modes of the class distributions. However, it is often difficult to obtain various mode (or pose) images of one person.

Support vector machine (SVM) based on kernels has been successfully applied for nonlinear classification problems such as face detection [29, 30]. However, this



(a) LDA (b) LDA mixture (c) LLDA

Figure 1: Comparison of LDA, LDA mixture and LLDA for the non-linear classification problem. Only LLDA guarantees that the two multi-modally distributed face classes in the input space are transformed into the class-wise single-modally distributed in the output space. Each upper plot shows the simulated data distributions and the components found by LDA, LDA mixture and LLDA. In the lower graphs the transformed class distributions in the global output coordinate system are drawn. The data is generated by $C_{11} = \{X \sim N(21.6, 2), Y \sim N(21.6, 1)\}$, $C_{12} = \{X \sim N(7.5, 2), Y \sim N(7.5, 0.8)\}$, $C_{21} = \{X \sim N(26, 2), Y \sim N(16, 2)\}$, and $C_{22} = \{X \sim N(8, 2), Y \sim N(16, 1.2)\}$,where $N(a, b)$ is a normal variable. 200 data points with mean a and standard deviation b are drawn for each mode. C_{ij} is the j-th cluster of the i-th class, u_{ij} is the j-th component of the i-th cluster and u_i denotes the i-th component of the output coordinate system.

is inefficient for multi-class recognition and inappropriate when a single sample per class is available to build a class model. By design generalized discriminant analysis (GDA) [2, 14, 22, 31] is suitable for multi-class face recognition problems whereby the original data is mapped into a high-dimensional feature space via a kernel function. The GDA representation learnt from training face classes of various pose images can be exploited to achieve pose robust representation of novel face classes. Therefore, recognition with a single model image of the novel classes is facilitated. However, GDA generally has the drawback of high computational cost in classification and overfitting. In applications such as classification of large data sets on the Internet or video, the computational complexity is particularly important. The global structure of nonlinear manifolds was represented by a locally linear structure in [5, 11]. These methods perform unsupervised learning for locally linear dimensionality reduction but not a supervised learning for discrimination.

In this study, several locally linear transformations are concurrently sought so that the class structures manifest by the locally transformed data are well sepa-

rated in the output space. The proposed method is called "Locally Linear Discriminant Analysis (LLDA)". The underlying idea of the proposed approach is that global nonlinear data structures are locally linear and local structures can be linearly aligned. Single class training objects, even if multi-modally distributed, are transformed into a cluster that is as small as possible with a maximum distance to the different class training objects, by a set of locally linear functions, as illustrated in Figure 1 (c). The linear functions learnt from training face classes of various pose images can be efficiently generalized to novel classes. Even when a single model image per class is provided, it is much easier to recognize a novel view image in the aligned output space.

The advocated method maximizes the separability of classes locally while promoting consistency between the multiple local representations of single class objects. Compared with the conventional nonlinear methods based on kernels, the proposed method is much more computationally efficient because it only involves linear transformations. By virtue of its linear base structure the proposed method also reduces overfitting normally exhibited by conventional non-linear methods. The transformation functions (or bases) learned from the face images of two different views are visualized in the Figure 2 (a). The functions can be exploited as the bases of a low dimensional subspace for robust face recognition. The basis functions of each cluster are specific to a particular facial pose. We note two interesting points in this Figure. First the bases of each cluster are similar to those of classical LDA and this ensures that face images of different identities at the same pose are discriminative. Secondly, the corresponding components of the two different clusters, for example, u_{f1} and u_{r1} are aligned to each other. They are characterized by a certain rotation and scaling with similar intensity variation. In consequence, face images of the same identity at different poses have quasi-invariant representation as shown in Figure 2 (a) and (b). For conciseness, only four face classes are plotted in the subspaces of Principal Component Analysis (PCA) [24], view-based LDA (or LDA mixture) and LLDA in Figure 2 (b). Each class has the four samples of two different poses and two different time sessions. While LDA and view-based LDA have shuffled class samples, LLDA achieves class distinctive distributions of samples.

The chapter is organized as follows: The next section briefly reviews the conventional methods for linear and nonlinear discriminant analysis. The proposed LLDA method is formulated in Section 3 and a solution of the optimization problem involved is presented in Section 4. Section 5 further simplifies the proposed method by replacing the Gaussian mixture model with the case that combines K-means clustering. Section 6 is devoted to the analysis of the computational complexity. Section 7.1 presents the results of experiments performed to demonstrate the beneficial properties of the proposed method on synthetic data. In Section 7.2, the method is applied to face recognition problem. Conclusions are drawn in Section 8.

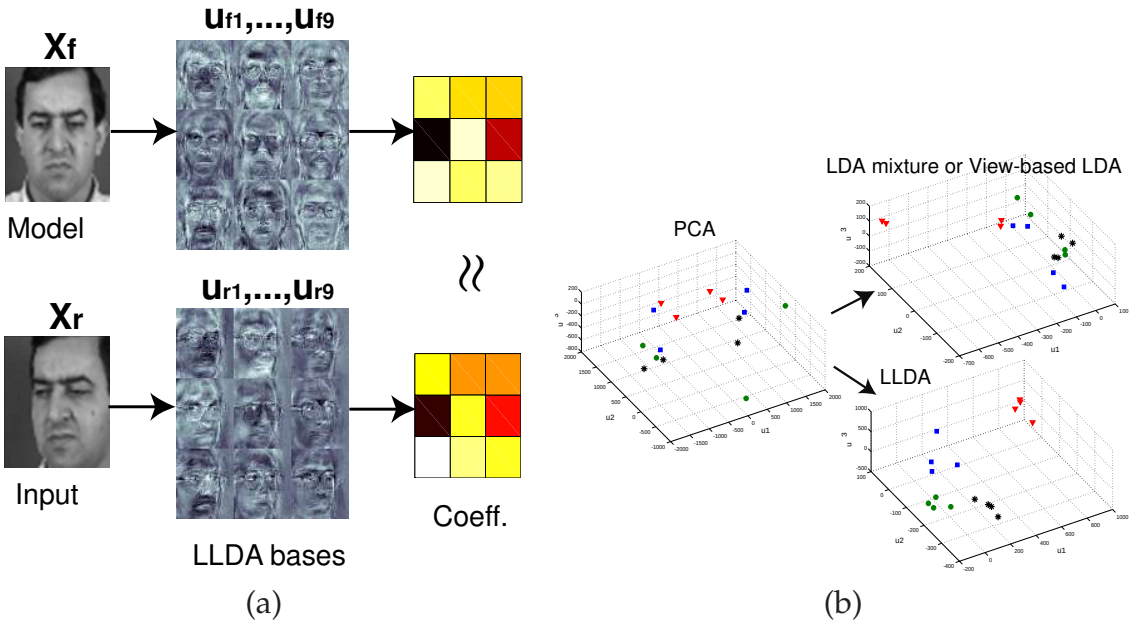


Figure 2: LLDA Representation. (a) Locally discriminative and aligned LLDA bases yield similar representations of posed face images. u_{ij} denotes the j-th component of the i-th cluster. (b) Face image distributions in the first three dimensions of PCA, view-based LDA and LLDA. Whereas LDA and view-based LDA have shuffled class samples, LLDA achieves class distinctive distributions. Different classes are marked as different symbols.

2 Review of Conventional Linear and Nonlinear Discriminant Methods

2.1 Linear Discriminant Analysis

LDA is a class specific method in the sense that it represents data to make it useful for classification [8]. Let $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_M\}$ be a data set of given N-dimensional vectors of face images. Each data point belongs to one of C object classes $\{\mathbf{X}_1, ..., \mathbf{X}_c, ..., \mathbf{X}_C\}$. The between-class scatter matrix and the within-class scatter matrix are defined as

$$\mathbf{B} = \sum_{c=1}^{C} M_c(\mathbf{m}_c - \mathbf{m})(\mathbf{m}_c - \mathbf{m})^T, \quad \mathbf{W} = \sum_{c=1}^{C} \sum_{\mathbf{x} \in \mathbf{X}_c} (\mathbf{x} - \mathbf{m}_c)(\mathbf{x} - \mathbf{m}_c)^T,$$

where $\mathbf{m}_c$ denotes the class mean and $\mathbf{m}$ is the global mean of the entire sample. The number of vectors in class $\mathbf{X}_c$ is denoted by M_c. LDA finds a matrix, $\mathbf{U}$, maximizing the ratio of the determinant of the between-class scatter matrix to the

determinant of the within-class scatter matrix as

$$\mathbf{U}_{opt} = \max_{arg\mathbf{U}} \frac{|\mathbf{U}^T\mathbf{B}\mathbf{U}|}{|\mathbf{U}^T\mathbf{W}\mathbf{U}|} = [\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N].$$

The solution $\{\mathbf{u}_i | i = 1, 2, ..., N\}$ is a set of generalized eigenvectors of $\mathbf{B}$ and $\mathbf{W}$ i.e., $\mathbf{B}\mathbf{u}_i = \lambda_i \mathbf{W}\mathbf{u}_i$. Usually PCA is performed first to avoid a singularity of the within-class scatter matrix commonly encountered in face recognition [20, 21].

2.2 Generalized Discriminant Analysis

The GDA [2] is a method designed for non-linear classification based on a kernel function Φ which transforms the original space $\mathbf{X}$ to a new high dimensional feature space $\mathbf{Z}$ s.t. $\Phi : \mathbf{X} \rightarrow \mathbf{Z}$. The within-class (or total) scatter and between-class scatter matrix of the non-linearly mapped data is

$$\mathbf{B}^\Phi = \sum_{c=1}^{C} M_c \mathbf{m}_c^\Phi (\mathbf{m}_c^\Phi)^T, \quad \mathbf{W}^\Phi = \sum_{c=1}^{C} \sum_{\mathbf{x} \in \mathbf{X}_c} \Phi(\mathbf{x}) \Phi(\mathbf{x})^T,$$

where $\mathbf{m}_c^\Phi$ is the mean of class $\mathbf{X}_c$ in $\mathbf{Z}$ and M_c is the number of samples belonging to $\mathbf{X}_c$. The aim of the GDA is to find such projection matrix $\mathbf{U}^\Phi$ that maximizes the ratio

$$\mathbf{U}_{opt}^\Phi = \max_{arg\mathbf{U}^\Phi} \frac{|(\mathbf{U}^\Phi)^T\mathbf{B}^\Phi\mathbf{U}^\Phi|}{|(\mathbf{U}^\Phi)^T\mathbf{W}^\Phi\mathbf{U}^\Phi|} = [\mathbf{u}_1^\Phi, ..., \mathbf{u}_N^\Phi].$$

The vectors, $\mathbf{u}^\Phi$ can be found as the solution of the generalized eigenvalue problem i.e. $\mathbf{B}^\Phi\mathbf{u}_i^\Phi = \lambda_i \mathbf{W}^\Phi \mathbf{u}_i^\Phi$. The training vectors are supposed to be centered (zero mean, unit variance) in the feature space $\mathbf{Z}$. From the theory of reproducing kernels, any solution $\mathbf{u}^\Phi \in \mathbf{Z}$ must lie in the span of all training samples in $\mathbf{Z}$, i.e.,

$$\mathbf{u}^\Phi = \sum_{c=1}^{C} \sum_{i=1}^{M_c} \alpha_{ci} \Phi(\mathbf{x}_{ci}),$$

where α_{ci} are some real weights and $\mathbf{x}_{ci}$ is the i-th sample of class c. The solution is obtained by solving

$$\lambda = \frac{\boldsymbol{\alpha}^T\mathbf{K}\mathbf{D}\mathbf{K}\boldsymbol{\alpha}}{\boldsymbol{\alpha}^T\mathbf{K}\mathbf{K}\boldsymbol{\alpha}},$$

where $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_c), c = 1, ..., C$ is a vector of weights with $\boldsymbol{\alpha}_c = (\alpha_{ci}), i = 1, , M_c$. The kernel matrix $K(M \times M)$ is composed of the dot products of non-linearly mapped data, i.e.

$$\mathbf{K} = (\mathbf{K}_{kl})_{k=1,...,C,\ l=1,...,C},$$

where $\mathbf{K}_{kl} = (k(\mathbf{x}_{ki}, \mathbf{x}_{lj}))_{i=1,...,M_k,\ j=1,...,M_l}$. The matrix $D(M \times M)$ is a block diagonal matrix such that

$$\mathbf{D} = (\mathbf{D}_c)_{c=1,...,C},$$

where c-th matrix $\mathbf{D}_c$ on the diagonal has all elements equal to $1/M_c$. Solving the eigenvalue problem yields the coefficient vectors $\boldsymbol{\alpha}$ that define the projection vectors $\mathbf{u}^{\Phi} \in \mathbf{Z}$. A projection of a testing vector $\mathbf{x}_{test}$ is computed as

$$(\mathbf{u}^{\Phi})^T \Phi(\mathbf{X}_{test}) = \sum_{c=1}^{C} \sum_{i=1}^{M_c} \boldsymbol{\alpha}_{ci} k(\mathbf{x}_{ci}, \mathbf{x}_{test}).$$

3 Locally Linear Discriminant Analysis (LLDA)

The proposed method, LLDA is applicable to multi-class nonlinear classification problems by using a set of locally linear transformations. Similarly to the notation adopted in Section 2, consider a data set $\mathbf{X} = \{\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_M\}$ of N-dimensional vectors of face images and C classes $\{\mathbf{X}_1, ..., \mathbf{X}_c, ..., \mathbf{X}_C\}$. The input vectors are clustered into K subsets denoted by $k,\ k = 1, ..., K$ and each subset k represents a cluster which a different transformation function is applied to. A cluster is defined by K-means clustering or Gaussian mixture modelling of the input vectors. The number of clusters K is chosen to maximize an objective function defined on the training set. Because K usually is a small positive integer, we can make the best choice of K empirically. Assuming that the multi-modality of the face data distribution is caused by the different poses, it is also pertinent to select K as the number of pose groups. However, general model order selection for a high dimensional data set remains an open problem. The basic LLDA approach draws on the notion of 'soft clustering', in which each data point belongs to each of the clusters with a posterior probability $P(k|\mathbf{x})$. The algorithm, that is combined with 'hard' K-means clustering, will be discussed in Section 5. We define the locally linear transformation $\mathbf{U}_k = [\mathbf{u}_{k1}, \mathbf{u}_{k2}, ..., \mathbf{u}_{kN}],\ k = 1, ..., K$ such that

$$\mathbf{y}_i = \sum_{k=1}^{K} P(k|\mathbf{x}_i) \mathbf{U}_k^T (\mathbf{x}_i - \boldsymbol{\mu}_k), \tag{1}$$

where N is the dimension of the transformed space. The mean vector of the k-th cluster $\boldsymbol{\mu}_k$ is described by

$$\boldsymbol{\mu}_k = \left(\sum_{i=1}^{M} P(k|\mathbf{x}_i) \mathbf{x}_i \right) / \left(\sum_{i=1}^{M} P(k|\mathbf{x}_i) \right). \tag{2}$$

The locally linear transformation matrices $\mathbf{U}_k$ are concurrently found so as to maximize the criterion function, J. Two objective functions are considered,

$$J_1 = \log(|\widetilde{\mathbf{B}}|/|\widetilde{\mathbf{W}}|), \text{ and } J_2 = (1-\alpha)|\widetilde{\mathbf{B}}| - \alpha|\widetilde{\mathbf{W}}|, \tag{3}$$

where $\widetilde{\mathbf{B}}$ and $\widetilde{\mathbf{W}}$ are the between-class and within-class scatter matrices in the locally linear transformed feature space respectively. The constant α takes values from the interval [0 1]. The objective functions maximize the between-class scatter while minimizing the within-class scatter in the locally transformed feature space. One of the differences between the two defined objective functions is manifest in the efficiency of "learning". The log objective function J_1 has the benefit of not requiring a free parameter α but it is more costly computationally. The function J_2 can efficiently be optimised iteratively, once α is selected. This is exemplified in the subsequent section. In terms of their performance, the two approaches are similar as reported in the experimental section 7.1. The global mean $\widetilde{\mathbf{m}}$ of all the transformed samples is

$$\widetilde{\mathbf{m}} = \frac{1}{M}\sum_{i=1}^{M}\mathbf{y}_i = \frac{1}{M}\sum_{i=1}^{M}\sum_{k=1}^{K}P(k|\mathbf{x}_i)\mathbf{U}_k^T(\mathbf{x}_i - \boldsymbol{\mu}_k), \tag{4}$$

where M is the total number of the samples. By substituting for $\boldsymbol{\mu}_i$ from equation (2), we get $\widetilde{\mathbf{m}} = \overrightarrow{0}$. The sample mean for class c which consists of M_c samples is given by

$$\widetilde{\mathbf{m}}_c = \frac{1}{M_c}\sum_{\mathbf{x}\in\mathbf{X}_c}\mathbf{y} = \sum_{k=1}^{K}\mathbf{U}_k^T\mathbf{m}_{ck}, \tag{5}$$

where $\mathbf{m}_{ck} = \frac{1}{M_c}\sum_{\mathbf{x}\in\mathbf{X}_c}P(k|\mathbf{x})(\mathbf{x}-\boldsymbol{\mu}_k)$.

The term $\mathbf{m}_{ck}$ denotes the sample mean of a class c in the k-th cluster. Because the transformation is defined with respect to the original cluster mean $\boldsymbol{\mu}_k$, the total mean $\widetilde{\mathbf{m}}_k$ of the transformed data in every cluster becomes zero. Using equations (4) and (5) the transformed between-class scatter matrix is given as:

$$\begin{aligned}
\widetilde{\mathbf{B}} &= \sum_{c=1}^{C}M_c(\widetilde{\mathbf{m}}_c - \widetilde{\mathbf{m}})(\widetilde{\mathbf{m}}_c - \widetilde{\mathbf{m}})^T \\
&= \sum_{c=1}^{C}M_c\left(\sum_{k=1}^{K}\mathbf{U}_k^T\mathbf{m}_{ck}\right)\left(\sum_{k=1}^{K}\mathbf{U}_k^T\mathbf{m}_{ck}\right)^T \\
&= \sum_{k=1}^{K}\mathbf{U}_k^T\mathbf{B}_k\mathbf{U}_k + \sum_{i=1}^{K-1}\sum_{j=i+1}^{K}\mathbf{U}_i^T\mathbf{B}_{ij}\mathbf{U}_j + \left(\sum_{i=1}^{K-1}\sum_{j=i+1}^{K}\mathbf{U}_i^T\mathbf{B}_{ij}\mathbf{U}_j\right)^T
\end{aligned} \tag{6}$$

where

$$\mathbf{B}_k = \sum_{c=1}^{C} M_c \mathbf{m}_{ck} \mathbf{m}_{ck}^T, \text{ and } \mathbf{B}_{ij} = \sum_{c=1}^{C} M_c \mathbf{m}_{ci} \mathbf{m}_{cj}^T.$$

The between-class scatter matrix consists of the scatter matrices associated with the respective clusters and the correlation matrix of the data samples belonging to two different clusters. The correlation matrix encodes the relationships of the two local structures for alignment. Similarly, the within-class scatter is defined by

$$\begin{aligned} \widetilde{\mathbf{W}} &= \sum_{c=1}^{C} \sum_{\mathbf{x} \in \mathbf{X}_c} (\mathbf{y} - \widetilde{\mathbf{m}}_c)(\mathbf{y} - \widetilde{\mathbf{m}}_c)^T \qquad (7) \\ &= \sum_{k=1}^{K} \mathbf{U}_k^T \mathbf{W}_k \mathbf{U}_k + \sum_{i=1}^{K-1} \sum_{j=i+1}^{K} \mathbf{U}_i^T \mathbf{W}_{ij} \mathbf{U}_j + \left(\sum_{i=1}^{K-1} \sum_{j=i+1}^{K} \mathbf{U}_i^T \mathbf{W}_{ij} \mathbf{U}_j \right)^T, \end{aligned}$$

$$\mathbf{W}_k = \sum_{c=1}^{C} \sum_{\mathbf{x} \in \mathbf{X}_c} \left(P(k|\mathbf{x})(\mathbf{x} - \boldsymbol{\mu}_k) - \mathbf{m}_{ck} \right) \left(P(k|\mathbf{x})(\mathbf{x} - \boldsymbol{\mu}_k) - \mathbf{m}_{ck} \right)^T$$

$$\mathbf{W}_{ij} = \sum_{c=1}^{C} \sum_{\mathbf{x} \in \mathbf{X}_c} \left(P(i|\mathbf{x})(\mathbf{x} - \boldsymbol{\mu}_i) - \mathbf{m}_{ci} \right) \left(P(j|\mathbf{x})(\mathbf{x} - \boldsymbol{\mu}_j) - \mathbf{m}_{cj} \right)^T.$$

Matrix $\mathbf{W}_k$ describes a local cluster and $\mathbf{W}_{ij}$ is the cross-term of two local clusters.

Generalization. Please note that the proposed algorithm without the cross terms $\mathbf{B}_{ij}$ and $\mathbf{W}_{ij}$ would adhere to the same concept as the LDA mixture model by focusing just on the local separability. Moreover, the defined criterion with $K = 1$ is identical to that of the conventional LDA.

4 Gradient based Solution for LLDA

In this section, we provide an efficient iterative optimisation method based on a gradient learning algorithm for an optimal set of locally linear transformation functions. While it is hard to find good parameters of a kernel function for new data in the conventional GDA, the proposed learning has only parameters which reduce or eliminate overfitting. The discriminant based on such a piecewise linear structure has the benefit of optimising a convex function with respect to the set of basis vectors of the local coordinates, yielding a unique maximum.

The method is based on a one-basis vector solution for $\mathbf{u}_{k1}$, $k = 1, ..., K$. Other methods based on incremental one-basis at a time solution can be found in [1, 33, 34] for discriminant or independent component analysis criteria. The proposed gradient method yields a global maximum solution by virtue of the criterion func-

tion being 2nd-order convex with respect to all the variables $\mathbf{u}_{k1}$, $k = 1, ..., K$. We need to run the one-basis algorithm several times to obtain a multidimensional solution $\mathbf{U}_k = [\mathbf{u}_{k1}, \mathbf{u}_{k2}, ..., \mathbf{u}_{kN}]$, $k = 1, ..., K$. The vector orthogonalization is performed to prevent different vectors from converging to the same maxima in every iteration. We seek the vectors $\mathbf{u}$ which maximize the criterion function under the constraint of being unit norm vectors:

$$\begin{aligned} &\text{Max } J_1 \text{ or } J_2, \\ &\text{for } ||\mathbf{u}_{kn}|| = 1, k = 1, ..., K \text{ and } n = 1, ..., N. \end{aligned} \tag{8}$$

This constrained optimization problem is solved by the method of projections on the constraint set [1]. A vector normalization imposing a unit norm is executed after every update of the vector. The learning rules are as follows:

Do the following steps with an index n starting from 1 to N for $\mathbf{u}_{kn}$, $k = 1, ..., K$.

1. Randomly initialize K unit vectors $\mathbf{u}_{kn}$.

2. Calculate the gradient of the objective function with respect to the variables $\mathbf{u}_{kn}$ by

$$\frac{\partial J_1}{\partial \mathbf{u}_{kn}} = \left(2\widetilde{\mathbf{B}}^{-1}\mathbf{B}_k - 2\widetilde{\mathbf{W}}^{-1}\mathbf{W}_k\right)\mathbf{u}_{kn} + \sum_{i=1, i\neq k}^{K}\left(2\widetilde{\mathbf{B}}^{-1}\mathbf{B}_{ki} - 2\widetilde{\mathbf{W}}^{-1}\mathbf{W}_{ki}\right)\mathbf{u}_{in}, \text{or}$$

$$\frac{\partial J_2}{\partial \mathbf{u}_{kn}} = \left(2(1-\alpha)\mathbf{B}_k - 2\alpha\mathbf{W}_k\right)\mathbf{u}_{kn} + \sum_{i=1, i\neq k}^{K}\left(2(1-\alpha)\mathbf{B}_{ki} - 2\alpha\mathbf{W}_{ki}\right)\mathbf{u}_{in}. \tag{9}$$

3. Update with an appropriate step size η as

$$\Delta\mathbf{u}_{kn} \leftarrow \eta\frac{\partial J}{\partial \mathbf{u}_{kn}}. \tag{10}$$

4. Carry out the deflationary orthogonalization by

$$\mathbf{u}_{kn} \leftarrow \mathbf{u}_{kn} - \sum_{i=1}^{n-1}\left(\mathbf{u}_{kn}^T\mathbf{u}_{ki}\right)\mathbf{u}_{ki}. \tag{11}$$

5. Normalize the vectors $\mathbf{u}_{kn}$ by

$$\mathbf{u}_{kn} \leftarrow \mathbf{u}_{kn}/||\mathbf{u}_{kn}||. \tag{12}$$

Repeat the processes $2 \sim 5$ until the algorithm converges to a stable point, set $n := n + 1$ and then go to the step 1.

Note that the two objective functions have different learning costs. When calculating the gradients of J_2 in (9), all the matrices, here scalar values, are previously given but the two matrices $\widetilde{\mathbf{B}}^{-1}, \widetilde{\mathbf{W}}^{-1}$ in the learning of J_1 should be iteratively updated. For the synthetic data example given in Figure 1, the optimization of J_1 takes about 15 times longer than that of J_2. While the learning of J_1 has a benefit of avoiding a free parameter α, J_2 has a simpler optimization cost when the parameter α is fixed. By changing α, one can control the importance of the variance of the between-class to that of the within-class data distributions. The orthogonalization (11) ensures that the proposed discriminant is defined by orthonormal basis vectors in each local coordinate system. The orthonormalisation of the bases yields more robust performance in the presence of estimation error (please refer to [33, 34] for the details). The benefits of orthonormal bases in discriminant analysis over the classical LDA have also been explained in the previous studies. Although we do not provide a proof of convergence or uniqueness of the gradient based iterative learning method, its convergence to a global maximum can be expected by virtue of the criterion being a 2nd-order convex function with respect to a basis vector, $\mathbf{u}_{kn}$, of each local coordinate system, and the joint set of the basis vectors $\mathbf{u}_{kn}$, $k = 1, ..., K$, as explained in [3, 17]. Figure 3 shows the convergence characteristics of the learning process for the synthetic data presented in Figure 1. The constant α was explored in steps of 0.1 and 0.1 was found to maximize the value of J_2. The value of J_2 according to the angles of basis vectors has a unique global maximum. It is also noted that the gradient optimization method of the objective function quickly converges regardless of constant α. The learning using the objective function J_1 also stably approaches a unique maximum.

Lagrangian method for the constrained optimization. A solution to the constrained optimization problem can also be obtained by using the method of Lagrangian multipliers as

$$L = (1 - \alpha)|\widetilde{\mathbf{B}}| - \alpha|\widetilde{\mathbf{W}}| - \sum_{k=1}^{K} \Lambda_k (\mathbf{U}_k^T \mathbf{U}_k - \mathbf{I}), \tag{13}$$

where $\mathbf{I}$ is the identity matrix and the diagonal matrix of eigen-values is

$$\Lambda_k = \begin{bmatrix} \lambda_{k1} & & \mathbf{0} \\ & \ddots & \\ \mathbf{0} & & \lambda_{kN} \end{bmatrix}$$

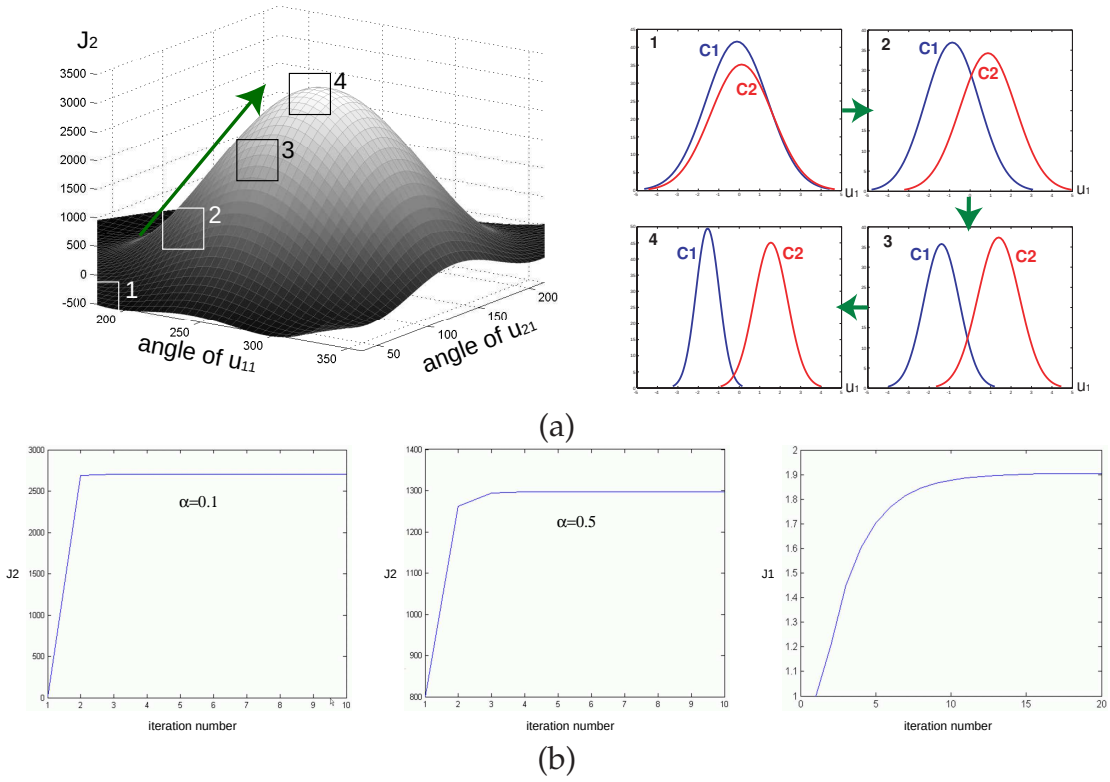


Figure 3: Convex optimization in LLDA learning. The proposed gradient-based learning is performed for the data distribution shown in Figure 1, where K is set to 2 and step size η is fixed to 0.1. (a) Value of the criterion J_2 (left) as a function of orientation of $\mathbf{u}_{11}, \mathbf{u}_{21}$ with $\alpha = 0.1$. The distributions of the two classes $C_1 = C_{11} \bigcup C_{12}, C_2 = C_{21} \bigcup C_{22}$ on the first major component $\mathbf{u}_1$, are drawn (right) as a series while J_2 is maximized. (b) Convergence graphs of J_2 with $\alpha = 0.1, 0.5$ and J_1.

The gradient of the Lagrangian function with respect to the basis vectors is

$$\frac{\partial L}{\partial \mathbf{u}_{kn}} = \left(2(1-\alpha)\mathbf{B}_k - 2\alpha\mathbf{W}_k - 2\lambda_{kn}\mathbf{I}\right)\mathbf{u}_{kn} + \sum_{i=1, i\neq k}^{K} \left(2(1-\alpha)\mathbf{B}_{ki} - 2\alpha\mathbf{W}_{ki}\right)\mathbf{u}_{in} = 0 \tag{14}$$

The solution can be found by numerical optimization of the Lagrangian function. However, in practice, a numerical optimization can only be used in low dimensional data spaces. As a reference, we utilized the numerical optimization "solve" function in Matlab to solve the two dimensional problem shown in Figure 1. The constraint optimization took 600 times longer than the gradient based optimization of J_2. The two proposed methods of gradient based learning are much favoured for their efficiency.

5 LLDA with K-means Clustering

Let us revisit the basic model derived in Section 3 by considering the special case involving a discrete posterior probability. K-means clustering divides a data set into disjoint subsets. If the data point $\mathbf{x}$ belongs to the k^*-th cluster, $P(k^*|\mathbf{x}) = 1$ and $P(k|\mathbf{x}) = 0$ for all the other k's. The mean vector of the k-th cluster $\boldsymbol{\mu}_k$ in (2) can be rewritten by

$$\boldsymbol{\mu}_k = \left(\sum_{\mathbf{x}} P(k|\mathbf{x})\mathbf{x}\right) / \left(\sum_{\mathbf{x}} P(k|\mathbf{x})\right) = \left(\sum_{\mathbf{x}\in k} \mathbf{x}\right) / M'_k, \tag{15}$$

where M'_k is the sample number of the cluster k. The defined transformation in (1) becomes

$$\mathbf{y} = \mathbf{U}_k^T(\mathbf{x} - \boldsymbol{\mu}_k) \text{ for } \mathbf{x} \in k. \tag{16}$$

The definition of the global mean (4) and the class mean (5) changes as follows:

$$\widetilde{\mathbf{m}} = \frac{1}{M}\sum_{k=1}^{K} \mathbf{U}_k^T \sum_{\mathbf{x}\in k}(\mathbf{x} - \boldsymbol{\mu}_k) = \overrightarrow{0}, \qquad \widetilde{\mathbf{m}}_c = \sum_{k=1}^{K} \mathbf{U}_k^T \mathbf{m}_{ck}, \tag{17}$$

where

$$\mathbf{m}_{ck} = \frac{1}{M_c} \sum_{\mathbf{x}\in \mathbf{X}_c \bigcap k} (\mathbf{x} - \boldsymbol{\mu}_k).$$

The transformed between-class matrix (6) and the within-class scatter matrix (7) can similarly be expressed by changing the notation from $P(k|\mathbf{x})$ to $\mathbf{x} \in k$. The learning algorithm in Section 4 finds the optimal set of locally linear transformation $\mathbf{U}_k,\ k = 1, ..., K$.

When a new pattern $\mathbf{x}_{test}$ is presented, it is first assigned to one of the clusters by

$$\mathbf{x}_{test} \in k^* = \min_{argk} ||\mathbf{x}_{test} - \boldsymbol{\mu}_k|| \tag{18}$$

and transformed by using the corresponding function

$$\mathbf{y}_{test} = \mathbf{U}_{k^*}^T(\mathbf{x}_{test} - \boldsymbol{\mu}_{k^*}). \tag{19}$$

6 Computational Complexity

The complexity of the algorithms depends on the computational costs associated with extracting the features and with matching.

Feature extraction. For the linear subspace methods such as PCA and LDA, the cost of feature extraction is determined by the dimensionality N of the input vector, $\mathbf{x}$, and the number of components of the subspace S. The cost of extracting features using linear methods is approximately proportional to $N \times S$. In the nonlinear subspace methods like the GDA, the n-th component of the projection of vector $\mathbf{x}$ is computed as

$$\mathbf{y}_n = \sum_{i=1}^{M} \alpha_{ni} k(\mathbf{x}_i, \mathbf{x}), \tag{20}$$

where M is the total number of training patterns, α_{ni} is a real weight and k denotes a kernel function. The cost of extracting features of the GDA is about $N \times S \times M$. The proposed method, LLDA has a similar cost with that of PCA or LDA depending on the preceding clustering algorithm. When a hard clustering such as K-means is applied, the cost of extracting features is $N \times (S + K)$, where the additional term $N \times K$ is for assigning a cluster to the input. When a soft clustering is applied, the cost is multiplied by the number of clusters, i.e., $N \times S \times K$. Note that usually $K \ll M$.

Matching. When the data points are represented as the S dimensional feature vectors and C gallery samples are given for the C class categories, the matching cost for recognition is $C \times S$. This applies to all, the linear, nonlinear and the proposed subspace methods.

7 Experiments

7.1 Results on Synthetic Data

Two sets of 2-dimensional synthetic data were experimented with. Set 1 has three classes which have two distinct modes in their distributions generated respectively by

$$\mathbf{X}_1 = \{X \sim N(7, 0.9), Y \sim N(4.1, 0.8)\} \bigcup \{X \sim N(-8.4, 0.9), Y \sim N(-3, 0.7)\}$$

$$\mathbf{X}_2 = \{X \sim N(5, 0.9), Y \sim N(0.1, 1)\} \bigcup \{X \sim N(-4, 0.9), Y \sim N(0.1, 0.6)\}$$

$$\mathbf{X}_3 = \{X \sim N(2.9, 0.9), Y \sim N(2.9, 0.5)\} \bigcup \{X \sim N(-4.2, 0.9), Y \sim N(-4.2, 0.4)\}$$

, where $N(a, b)$ is a normal variable which has a mean a and standard deviation b. 200 data points were drawn from each Gaussian mode. Set 2 has two classes which

have three distinct peaks in the distributions generated by

$$\mathbf{X}_1 = \{X \sim N(4.4, 1), Y \sim N(5.4, 0.5)\} \bigcup \{X \sim N(-4.7, 1), Y \sim N(-3.9, 0.2)\}$$

$$\bigcup \{X \sim N(4.4, 1), Y \sim N(-7.8, 0.8)\}$$

$$\mathbf{X}_2 = \{X \sim N(7.6, 1), Y \sim N(2.1, 0.9)\} \bigcup \{X \sim N(-5, 1), Y \sim N(-0.9, 0.6)\}$$

$$\bigcup \{X \sim N(1.6, 1), Y \sim N(-9.9, 0.7)\}$$

Conventional LDA, mixture of LDA, and GDA with the radial basis function (RBF) as a kernel are compared with LLDA in terms of classification error. Euclidean distance(E.D.), normalized correlation(N.C.) and Mahalanobis distance(M.D.) were utilized as similarity functions for the nearest neighbor (N.N.) classification. It is noted that all the transformed data points were compared with the sample mean of each class in (5).

In the method of LLDA, the number of clusters, K, was selected to maximize the value of the objective function. For the example of the data of Set 1, the peak values of J_1 changed with K as follows: -7.14, 2.97, 0.85 for $K = 1, 2, 3$ respectively, so the number $K = 2$ was chosen. This is much simpler than the parameter selection of RBF as a kernel function in GDA, because the standard deviation of RBF is hard to initialize and it is a real (non integer) value. The axes of LDA, LDA mixture, LLDA are drawn in Figure 4. Table 1 shows the average number of classification errors with their standard deviation and the relative costs of feature extraction. It is apparent that the proposed discriminant can well solve the non-linear classification problem on which the conventional linear methods fail and it is much profitable in terms of computational efficiency as compared to GDA. The feature extraction complexity of the proposed method is about 1/270 of that of GDA in this example. Although the accuracy of GDA was slightly better, it is noted that the kernel parameter of RBF in GDA was exhaustively searched to find the best performance for the given data. In contrast, the proposed algorithm based on the log objective function has only a small integer K to be adjusted and the learning process is also much faster. Additionally note that, when the class distributions have a single mode, LLDA with $K = 1$ yields a successful separation by behaving like the conventional LDA. LLDA with $K = 1$ is identical to the conventional LDA with the exception of the orthonormal constraint imposed on the axes by LLDA.

7.2 View-invariant Face Recognition with One Sample Image

The proposed algorithm has been validated on the problem of free pose face recognition in the scenario when only a single frontal image of each class is available as a gallery image. To recognize a novel view face, some prior experiences of face view changes are required. Conventional discriminative subspace methods such

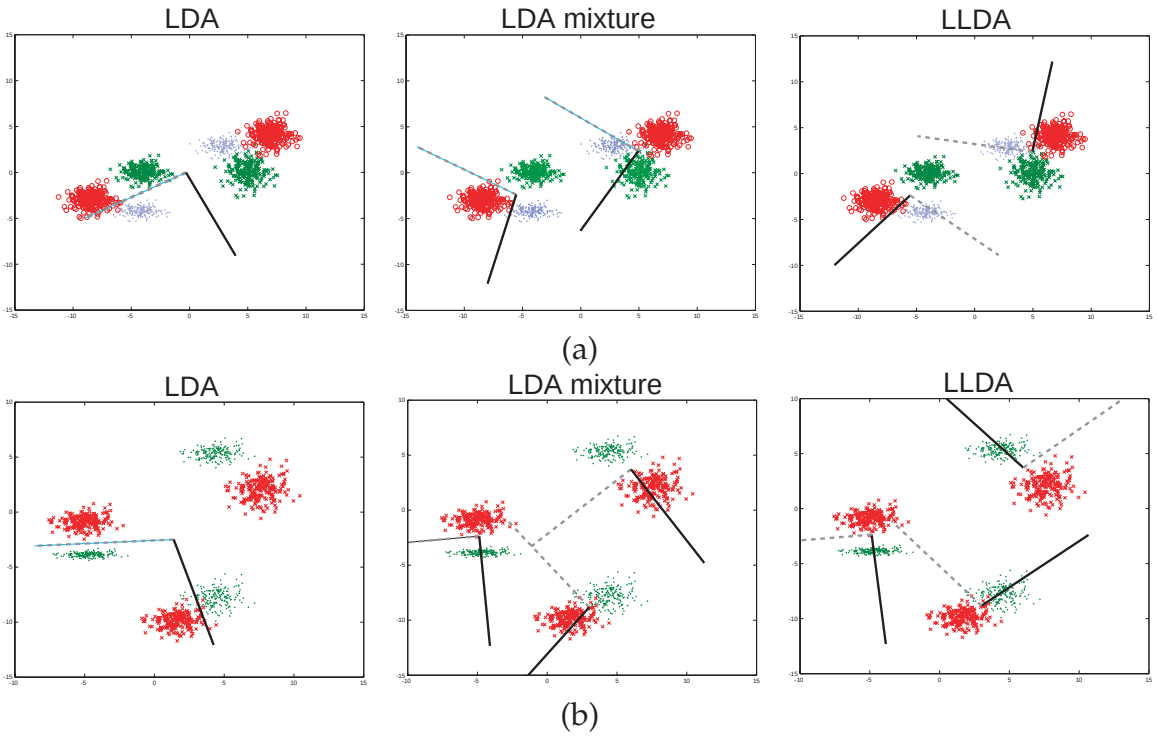


Figure 4: Simulated data distributions and the components found. Black solid lines represent the first major components and gray dashed lines the second components. (a) For Set 1. (b) For Set 2.

as LDA and GDA can be applied to learn a robust representation from any prototype face set which exhibits different poses. GDA has a benefit of capturing any nonlinear manifolds of face pose changes. Then, the learned subspace representation can be applied to new test identities. In contrast, SVM which performs binary classification and requires a considerable number of training samples for each class, is completely inappropriate for this scenario.

There are a number of conventional techniques that have been developed for view-invariant face recognition [4, 6, 10, 12, 13, 15, 25, 26, 28]. In spite of successes of some approaches [6, 10, 12, 13, 26], they have an important drawback of requiring dense correspondences of facial features for image normalization or more than one model images. The step of correspondence solving or detection of abundant salient facial features, which is needed for separating the shape and texture components of face images in these methods, is usually difficult itself. Errors in correspondences seriously degrade the performance of the subsequent recognition methods as shown in [12]. In our experiments, the proposed algorithm, LLDA, is compared with PCA, LDA and GDA as the benchmark subspace methods that have been successfully applied to face recognition in the past and FaceIt(v.5.0), the commercial face recognition system from Identix. FaceIt ranked top overall in the Face Recognition Vendor Test 2000 and 2002 [**?**, 32].

Database: We used the XM2VTS data set annotated with pose labels of the face.

	E.D.	N.C.	M.D.	Cost
Set1 (400 samples/class)				
LDA	266±115	266±115	81±61	1
LDA mixture	254±27	255±23	169±45	1+ω
GDA	4.3±1.1	4.3±1.1	4.4±0.5	270
LLDA J_1 + km	7.6±3.5	7.6±3.5	7±3.4	1+ω
LLDA J_2 + km	7.6±3.5	8±3.6	7.3±3.7	1+ω
LLDA J_1 + GMM	7.6±3.5	8±3.6	7.3±3.7	2+ω
Lagran. J_2	7.6±3.2	8±2.6	7.3±2.8	1+ω
Set2 (600 samples/class)				
LDA	308±129	308±129	207±272	1
LDA mixture	205±1.4	205±1.4	206±7	1+ω
GDA	4±1.4	4±1.4	4±0	278
LLDA J_1 + km	9.5±3.5	9.5±3.5	7.5±3.5	1+ω
LLDA J_2 + km	8±1.4	8±1.4	7±2.8	1+ω

Table 1: Classification Results (number of errors). ω indicates the computational cost of deciding which cluster a new pattern belongs to. It is usually less than 1. 'LLDA J_1 +km' is the LLDA of the objective function J_1 with K-means clustering algorithm. 'LLDA J_1 + GMM' indicates the LLDA of the objective function J_1 with Gaussian mixture modelling. 'Lagrangian J_2' denotes a numerical solution of the Lagrangian formulation.

The face database consists of 2950 facial images of 295 persons with 5 pose variations and 2 different time sessions which have 5 months time elapse. The data set consists of 5 different pose groups (F,R,L,U,D) which are captured at frontal view, about ±30 horizontal rotations and ±20 vertical rotations. The two images of a pose group 'F' captured at different times are denoted by F1 and F2. This may be the largest public database that contains images of faces taken from different view points. The images were normalized to 46*56 pixel resolution with a fixed eye position and some normalized data samples are shown in Figure 5. The face set is partitioned into the three subsets: 1250 images of 125 persons, 450 images of 45 persons and 1250 face images of 125 persons for the training(Tr), evaluation(Ev) and test(Te) respectively. Please note that the three sets have different face identities. For the test of the commercial FaceIt system, the original images were applied to the system with the manual eye positions.

Protocol and Setting: The training set is utilized to learn the subspace representation of the conventional PCA/LDA/GDA methods and LLDA with K-means. For efficiency of learning, all of the algorithms were applied to the first 80 ($\lambda_{80}/\lambda_1 = 0.004$) eigenfeatures of the face images. Figure 6 shows the plots of eigenvalues and J_1 of LLDA as a function of dimensionality. The evaluation set is utilized to adjust the kernel parameter of GDA(an RBF kernel with an adjustable width) and the dimensionality of the output vectors for all the methods. The parameters are

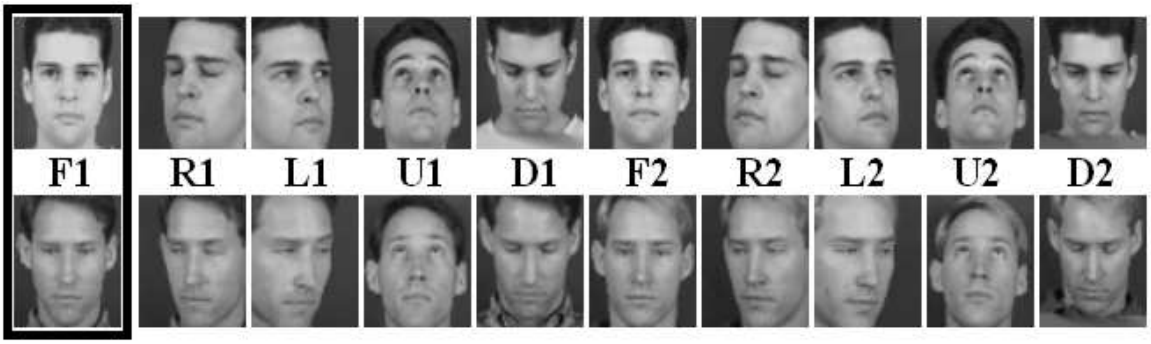
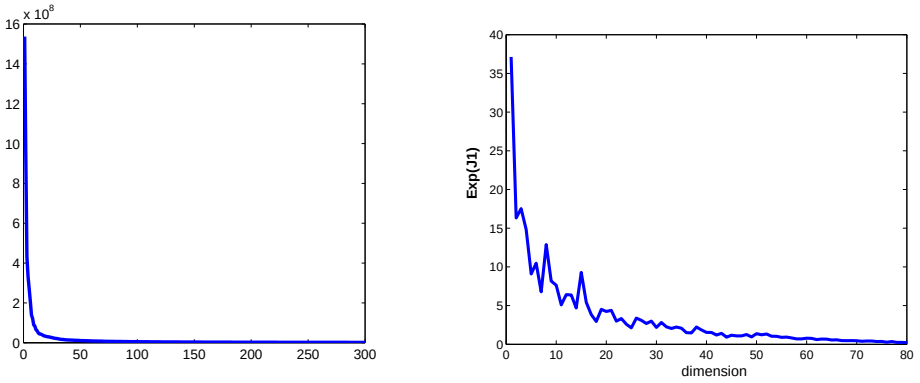


Figure 5: Normalized data samples. The left most image is given as the gallery image and other rotated face images are used as testing images.



(a) (b)

Figure 6: (a) Eigenvalues of the face data. (b) Plot of J1 as a function of dimensionality.

properly quantized and all combinations of the discrete values of the quantized parameters are examined to get the best recognition rate on the evaluation set. In LLDA, the number of clusters corresponded to the number of the pose groups and K-means algorithm was applied. The log objective function J_1 was utilized to learn the set of transformation functions and the learning rate was controlled to have faster convergence. Typically, the learning took 2 or 3 minutes in Pentium IV 2GHz PC.

In the test, the frontal face images of the test set, which are the leftmost images in Figure 5, are registered as a gallery and all the other images of the test set are exploited as queries. All the test images are projected into the learned subspace and Nearest-Neighbor based classification is performed based on the projection coefficients. Recognition rates in $(\%)$ are measured. In LLDA, test face images were assigned to one of the clusters by equation (18) and projected into the corresponding subspace by (19).

Results : Table 2 presents the recognition rates on the evaluation and test set and Figure 7 shows the performance curves of the test set as a function of dimensionality. The recognition rate of the evaluation and test set was much enhanced by

	PCA		LDA		GDA		LLDA		FaceIt	
	Ev	Te	Ev	Te	Ev	Te	Ev	Te	Ev	Te
R1	13	4	55	43	66	49	66	56	73	**64**
L1	8	8	55	45	77	57	73	**64**	66	52
U1	28	16	53	43	73	52	71	**66**	46	36
D1	33	29	68	55	84	**66**	75	60	37	24
F2	75	70	73	63	82	71	75	66	95	**83**
R2	8	3	42	22	46	29	40	35	46	**36**
L2	4	4	33	27	44	36	48	**47**	46	30
U2	17	15	28	28	35	35	40	**44**	24	23
D2	20	10	31	32	42	32	35	**40**	33	9
Avg.	23	18	49	40	61	47	58	**53**	51	39

Table 2: Face Recognition Rates (%).

the proposed algorithm. FaceIt exhibited the best recognition performance for the frontal images F2 but quite low recognition rates for the rotated faces especially involving up/down rotations. More results showing the effects of the elapsed time and the size of test population are given in Figure 8.

In LLDA, the number of clusters was chosen as the number of the pose groups as previously mentioned by assuming that the multi-modality of the face class distributions is caused by the different poses. In each cluster, classes are assumed to be linearly separable. Although this assumption is not necessarily true, as other factors such as time elapse can make a class distributed multi-modally and not linearly separable, we found that LLDA performed much better as compared with LDA/GDA/FaceIt. A performance degradation as a function of time was observed for all the methods but a relative performance gain exhibited by LLDA was still preserved as shown in Figure 8. As mentioned above, the results of the test set were obtained by utilizing the output dimensionality found to be the best for the evaluation set. The establishment of a proper evaluation set is important because the test results are sensitive to the output dimensionality as shown in Figure 7. This may be because the pose variation is so large that the methods find only few meaningful axes. We can see that the evaluation set used proved adequate to solve this peaking problem as the recognition results on the test set using the best dimensionality indicated by the evaluation set in Table 2 agreed with the best results of the graph in Figure 7. GDA had the tendency highly to overfit on the training set so that a separate evaluation set was needed to suppress this behaviour.

Regarding the complexity of the feature extraction, PCA, LDA and the LLDA are approximately identical and GDA about 40 times worse than the linear methods. Please note that the complexity of GDA depends on the size of the training set. The proposed method is not expensive in terms of computational costs and provides more robust and accurate performance for all the dimensionalities as compared with the other methods.

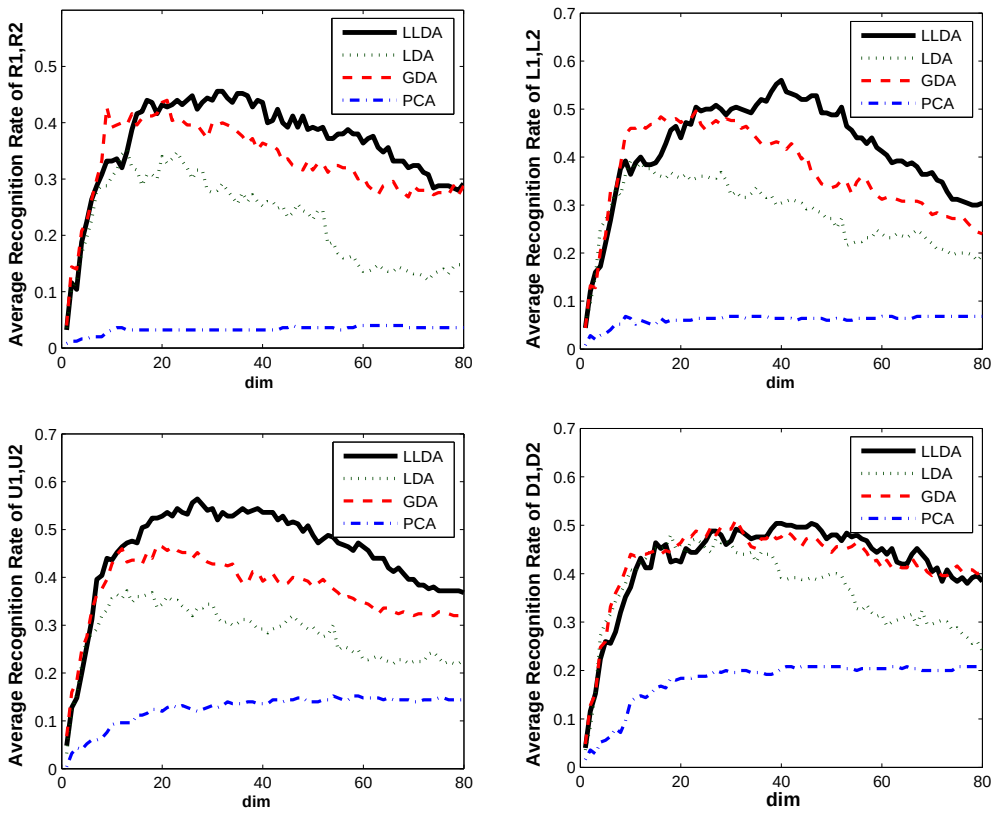
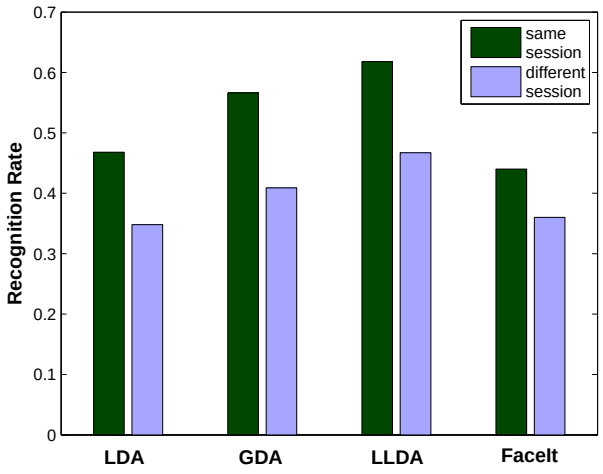


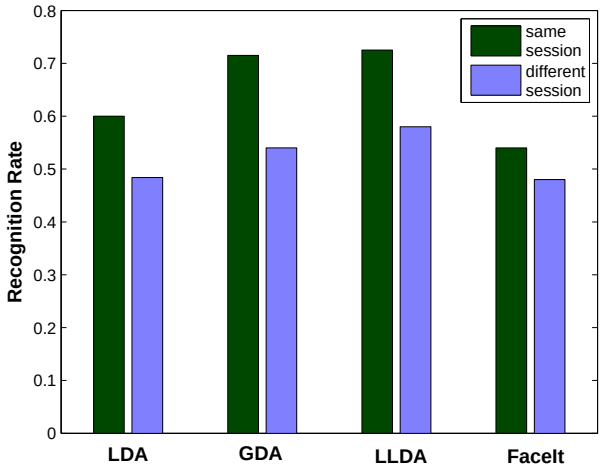
Figure 7: The test performance curves (in %)as a function of dimensionality.

8 Conclusion

A novel discriminant analysis method which can classify a non-linear structure has been proposed for face recognition. Face data set that exhibits large pose variations has nonlinear manifolds and is not linearly separable. A set of local linear transformations is found so that the locally linearly transformed classes maximize the between-class covariance and minimize the within-class covariance in a single global space. The proposed learning method for finding the optimal set of locally linear bases does not suffer from the local-maxima problem and stably converges to a global maximum point. The proposed discriminant provides a set of discriminant features for the view-invariant face recognition with a given single model image and it is highly efficient computationally as compared with the nonlinear discriminant analysis based on the kernel approach. By virtue of the linear base structure of the solution, the method reduces overfitting. We intend to improve the performance of the proposed approach by exploiting dense facial feature correspondences for an image regularization step in the future. The current performance was obtained with the images registered with a fixed eye position and this can be seen as a poor basis of the image normalization for the method. More elaborate regularization is expected to promote that face class structures are well separated by a set of local linear transformations similarly with the results of [4].



(a)



(b)

Figure 8: Recognition rates under aging for different sizes of test population. (a) Recognition rates on the test set consisting of 125 identities . (b) Recognition rates on the test set consisting of randomly chosen 50 identities.

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